

# QUADRICS VIA SEMIGROUPS

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ABSTRACT. Let  $M_2$  be the semigroup of linear endomorphisms of a plane. We show that the space of idempotents in  $M_2$  is a hyperboloid of one sheet, the set of semigroup-theoretic inverses of a nonzero singular element in  $M_2$  is a hyperbolic paraboloid, and the set of nilpotent elements in  $M_2$  is a right circular cone.

This is the story of the rediscovery of classical three-dimensional geometry, especially the geometry of quadric surfaces, while studying the semigroup  $M_2(\mathbb{R})$  of linear endomorphisms of a real plane. One of the surfaces that appears prominently in this context is the hyperboloid of one sheet, referred to as *spaghetti bundle* in [8]. In this story the spaghetti presents itself as the set of idempotents in  $M_2(\mathbb{R})$ , the cone emerges as the set of nilpotent elements and the hyperbolic paraboloid as the set of semigroup-theoretic inverses of a singular element.

This rediscovery was briefly announced in [5]. Generalizations of some of the ideas presented here to semigroups of linear endomorphisms of higher dimensional vector spaces are discussed in [6]. The little bit of semigroup theory quoted below is based on [3].

## 1. THE SEMIGROUP $M_2(\mathbb{R})$

A set  $S$  together with an associative binary operation in  $S$  is called a *semigroup*. An element  $e$  in  $S$  is called an *idempotent* if  $e^2 = e$ . If  $X \subseteq S$ , the set of idempotents in  $X$  is denoted by  $E(X)$ . In any semigroup  $S$  we can define certain equivalence relations, denoted by  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{J}$ ,  $\mathcal{D}$  and  $\mathcal{H}$ , and called Green's relations. Let  $S^1$  denote  $S$  if  $S$  has an identity element. Otherwise, let it denote  $S$  with an identity element 1 adjoined. For  $a, b \in S$  the first three are defined by

$$\begin{aligned} a \mathcal{L} b &\Leftrightarrow aS^1 = bS^1 \\ a \mathcal{R} b &\Leftrightarrow S^1a = S^1b \\ a \mathcal{J} b &\Leftrightarrow S^1aS^1 = S^1bS^1 \end{aligned}$$

and the remaining ones by  $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ ,  $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ .

If  $a \in S$ , the set of all elements in  $S$  which are  $\mathcal{L}$ -equivalent to  $a$  is denoted by  $L_a$  and is called the  $\mathcal{L}$ -class containing  $a$ . The notations  $R_a, J_a, D_a, H_a$  have similar meanings. These are the Green classes in  $S$ . Two elements  $a, a'$  in a semigroup  $S$  are called *inverse elements* if  $aa'a = a$ ,  $a'aa' = a'$ . A semigroup in which every element has an inverse is called a *regular semigroup*.

A well-known example of a regular semigroup is  $M_n(\mathbb{K})$  (where  $\mathbb{K} = \mathbb{R}$ , or  $\mathbb{K} = \mathbb{C}$ ) of linear endomorphisms of an  $n$ -dimensional vector space  $V$  over  $\mathbb{K}$  under

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composition of mappings. The set  $S_n$  of singular elements in  $M_n(\mathbb{K})$  is a regular subsemigroup of  $M_n(\mathbb{K})$ . Treating functions as right operators we see that, for  $a, b \in M_n(\mathbb{K})$ ,  $a \mathcal{L} b$  if and only if  $a$  and  $b$  have the same range, and,  $a \mathcal{R} b$  if and only if  $a$  and  $b$  have the same null space. Further,  $a \mathcal{D} b$  if and only if  $a$  and  $b$  have the same rank, and,  $a \mathcal{J} b$  is equivalent to  $a \mathcal{D} b$ .

As already indicated, the semigroup having special interest to us is  $M_2(\mathbb{R})$ , denoted by  $M_2$  in the sequel. We shall represent elements of  $M_2$  as square matrices of order 2 relative to some fixed ordered orthonormal basis for  $V$ . Listing out the entries in the elements of  $M_2$  row-wise we get vectors in  $\mathbb{R}^4$ . In this way we may identify  $M_2$  with  $\mathbb{R}^4$ . The usual inner product in  $\mathbb{R}^4$  can be represented using the trace function, defined by  $\text{tr}(x) = x_1 + x_4$ . If  $x = (x_1, \dots, x_4)$  and  $y = (y_1, \dots, y_4)$  then

$$\langle x, y \rangle = x_1 y_1 + \dots + x_4 y_4 = \text{tr}(x^* y).$$

where  $x^*$  is the transpose of  $x$ .

## 2. GEOMETRY OF THE GREEN CLASSES IN $M_2$

$M_2$  has three  $\mathcal{D}$ -classes, namely,  $D_0$ ,  $D_1$  and  $D_2$ , where  $D_k$  is the set of endomorphisms of rank  $k$ . Obviously we have  $D_0 = \{0\}$  which is simply a point. From the well-known fact (p.168 [1]) that the space  $M(m, n, k)$  of  $m \times n$  matrices of rank  $k$  is a manifold of dimension  $k(m + n - k)$  we immediately deduce that  $D_1$  is a three-dimensional submanifold of  $\mathbb{R}^4$ . What this means is that sufficiently small neighborhoods of every point in  $D_1$  'looks like' a three-dimensional euclidean space. Lastly  $D_2$  is the set  $\text{GL}(2)$  of all invertible elements in  $M_2$ . It is well known that  $\text{GL}(2)$  is a four-dimensional submanifold of  $\mathbb{R}^4$ .

To describe the  $\mathcal{L}$ - and  $\mathcal{R}$ -classes in  $M_2$ , we require some geometric terminology. A line in a linear space  $U$  is an affine subspace of  $U$  generated by two distinct points (that is, vectors) in  $U$  and a plane  $\mathcal{P}$  in  $U$  is an affine subspace generated by three non-collinear points in  $U$ . If  $\mathcal{P}$  passes through the origin in  $U$  then the set  $\mathcal{P} \setminus \{0\}$  is called a *punctured plane* in  $U$ . In a similar way we may define a *punctured line* in  $U$ .

If  $0 = a \in M_2$ , then  $L_a = R_a = \{0\}$ . Also, if  $a \in \text{GL}(2)$  then  $L_a = R_a = \text{GL}(2)$ . The classes  $L_a$  and  $R_a$ , when  $a \neq 0$  and  $a \notin \text{GL}(2)$ , are the nontrivial  $\mathcal{L}$ - and  $\mathcal{R}$ -classes in  $M_2$ . If  $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , a simple argument involving range and null space shows that

$$\begin{aligned} L_e &= \left\{ \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0) \right\}, \\ R_e &= \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0) \right\}. \end{aligned}$$

These are obviously punctured planes. That this is true for any  $0 \neq a \in S_2$  can be easily verified.

**Proposition 1.** *If  $0 \neq a \in S_2$ , then  $L_a$  and  $R_a$  are punctured planes lying in  $S_2$ .*

Surprisingly, the converse of Proposition 1 is also true, that is, every punctured plane in  $S_2$  is a nontrivial  $\mathcal{L}$ -class or  $\mathcal{R}$ -class in  $S_2$ . A nontrivial  $\mathcal{H}$ -class, being the intersection of a nontrivial  $\mathcal{L}$ -class and a nontrivial  $\mathcal{R}$ -class, is a punctured line in  $S_2$ .

3. INTERSECTION OF  $S_2$  WITH  $x_1 + x_4 = \lambda$ 

We have seen that  $D_1$  is a three-dimensional manifold sitting in  $\mathbb{R}^4$ . To know more about this manifold we consider intersections of  $S_2$  with hyperplanes in  $M_2$ . A hyperplane in  $M_2$  is the set of all  $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$  satisfying an equation of the form

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = \lambda$$

where  $a = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in M_2$ . This equation can be expressed in the form  $\text{tr}(ax) = \lambda$ . We denote this hyperplane by  $P(a; \lambda)$  and its intersection with  $S_2$  by  $SP(a; \lambda)$ . We first consider the special case  $SP(I; \lambda)$  where  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

**Proposition 2.** *If  $\lambda \neq 0$  then  $SP(I; \lambda)$  is a hyperboloid of one sheet. Also,  $SP(I; 0)$  is a right circular cone with vertex at the origin in  $M_2$ .*

*Proof.* Consider the following points in  $P(I; \lambda)$ :

$$O'(\frac{\lambda}{2}, 0, 0, \frac{\lambda}{2}), \quad A(\frac{\lambda}{2} + \frac{1}{\sqrt{2}}, 0, 0, \frac{\lambda}{2} - \frac{1}{\sqrt{2}}), \quad B(\frac{\lambda}{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\lambda}{2}), \quad C(\frac{\lambda}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\lambda}{2}).$$

Using the inner product and the induced norm in  $\mathbb{R}^4$ , we can see that the vectors  $\overrightarrow{O'A}$ ,  $\overrightarrow{O'B}$ ,  $\overrightarrow{O'C}$  form an orthonormal system in  $P(I; \lambda)$ . We choose  $O'$  as the origin and the directed lines joining  $O'$  to the points  $A, B, C$  as the coordinate axes. We shall refer to the coordinates of a point in  $P(I; \lambda)$  relative to this axes as the Bell-coordinates of the point.

Let the Bell-coordinates of any point  $Q(x_1, x_2, x_3, x_4)$  in  $P(I; \lambda)$  be  $(X, Y, Z)$  so that we have

$$\overrightarrow{O'P} = X\overrightarrow{O'A} + Y\overrightarrow{O'B} + Z\overrightarrow{O'C}.$$

Since  $\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OO'}$ , etc., we must have

$$(1) \quad x_1 = \frac{\lambda}{2} + \frac{X}{\sqrt{2}}, \quad x_2 = \frac{Y-Z}{\sqrt{2}}, \quad x_3 = \frac{Y+Z}{\sqrt{2}}, \quad x_4 = \frac{\lambda}{2} - \frac{X}{\sqrt{2}}.$$

If we substitute the above expressions in the equation defining the space  $S_2$ , namely,  $x_1x_4 - x_2x_3 = 0$ , we get

$$(2) \quad X^2 + Y^2 - Z^2 = \frac{\lambda^2}{2}.$$

If we consider any set of numbers  $(X, Y, Z)$  satisfying Eq.(2), then using the relations in Eq.(1), we can see that  $(X, Y, Z)$  are the Bell-coordinates of a point on  $SP(I; \lambda)$ . Therefore Eq.(2) is the equation of  $SP(I; \lambda)$  relative to the Bell-axes. When  $\lambda \neq 0$ , this equation represents a hyperboloid of one sheet (§64 [2]).

When  $\lambda = 0$ , Eq.(2) reduces to  $X^2 + Y^2 - Z^2 = 0$ , which represents a right circular cone with vertex at the origin and semi-vertical angle  $\frac{\pi}{4}$ . Note that this cone lies in the hyperplane  $P(I; 0)$ .  $\square$

By direct computation or otherwise one can easily see that  $SP(I; 0)$  is the set of nilpotent elements in  $M_2$ .

We now explore the relations between the geometrical properties of  $SP(I; \lambda)$  and the algebraic properties of  $M_2$ . Obviously the center of  $SP(I; \lambda)$  is  $O'$  which is a point on the line joining  $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since this line is independent of the choice of  $\lambda$ , the centers of the hyperboloids  $SP(I; \lambda)$ , for various values of  $\lambda$ , all lie on a fixed line. The axis of rotation of the hyperboloid is given by  $X = 0$ ,  $Y = 0$ . These equations produce a matrix of the form

$$\begin{bmatrix} \frac{\lambda}{2} & x_2 \\ -x_2 & \frac{\lambda}{2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{2} & 0 \\ 0 & \frac{\lambda}{2} \end{bmatrix} + \begin{bmatrix} 0 & x_2 \\ -x_2 & 0 \end{bmatrix}.$$

The set of all such elements form a line through the center of the hyperboloid parallel to the line formed by the set of skew-symmetric elements in  $M_2$ . The asymptotic cone of  $SP(I; \lambda)$  from its center (§78[2]) is  $X^2 + Y^2 - Z^2 = 0$ , which is independent of  $\lambda$ .

From the above discussion we can construct an image of  $S_2$ . If we imagine  $\lambda$  as ‘time’,  $S_2$  can be thought of as a space generated by the ‘moving’ hyperboloid  $SP(I; \lambda)$ . Initially, that is when  $\lambda = 0$ , we have a degenerate hyperboloid, namely, the right circular cone  $SP(I; 0)$  with vertex at  $O$  and axis the line  $\left\{ \begin{bmatrix} 0 & 0 \\ -\alpha & 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$ . As time advances numerically, the center of the hyperboloid moves along the line  $\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$ . The axis of rotation of the moving hyperboloid is a line through the center parallel to the axis of the ‘initial’ hyperboloid. Further, at time  $\lambda$ , the radius of the principal circular section of the hyperboloid is  $\frac{|\lambda|}{\sqrt{2}}$ . Clearly, this increases as  $\lambda$  increases numerically. Thus, the hyperboloid  $SP(I; \lambda)$  expands simultaneously as it moves away from the origin.

#### 4. THE SPACE OF IDEMPOTENTS

Let  $E(k)$  be the set of idempotents of rank  $k$ . Obviously,  $E(0) = \{0\}$  and  $E(2) = \{I\}$ . Also, it is easy to see that  $E(1) = SP(I; 1)$ . The next result follows from this.

**Proposition 3.** *The space  $E(1)$  of idempotents of rank 1 in  $M_2$  is a hyperboloid of one sheet.*

The generators of  $E(1)$  through  $e$  turn out to be the sets  $E(L_e)$  and  $E(R_e)$ . To see this it is enough to show that these sets are lines through  $e$ , for if a line lies wholly on a conicoid it must belong to a system of generating lines of the conicoid (§ 97 [2]). That  $E(L_e)$  and  $E(R_e)$  are lines follows from

$$E(L_e) = e + (1 - e)S_2e \quad E(R_e) = e + eS_2(1 - e),$$

which can be verified by direct computation.

Having found the generating lines on  $E(1)$ , we next determine how these are organized into two systems of generators as in [2]. Let  $\mathbf{L}_1$  be the family of all lines of the form  $E(L_e)$  and  $\mathbf{L}_2$ , the family of all lines of the form  $E(R_e)$ . Since  $V$  is 2-dimensional, no two distinct members of  $\mathbf{L}_1$  (or, of  $\mathbf{L}_2$ ) intersect. Also, every member of  $\mathbf{L}_1$  intersects every member (except one) of  $\mathbf{L}_2$ , and vice versa. Thus  $\mathbf{L}_1$  and  $\mathbf{L}_2$  are the two families of generating lines of  $E(1)$ . Though it is not an intrinsic property of  $E(1)$ , it is interesting to see that members of  $\mathbf{L}_1$  are members of the  $\lambda$ -system and those of  $\mathbf{L}_2$  are members of the  $\mu$ -system of generators of  $E(1)$  (in the terminology of [2]).

The equation of the plane of the principal circular section of  $SP(I; 1)$  is  $Z = 0$ , which is equivalent to  $x_2 = x_3$ , which, in turn, is equivalent to the statement that the element  $x \in M_2$  is symmetric. Thus the principal circular section of  $E(1)$  consists of the symmetric elements in  $E(1)$ .

#### 5. INTERSECTIONS OF $S_2$ WITH ARBITRARY HYPERPLANES

The first thing we notice is that the coordinates  $(X, Y, Z)$  of any point on  $SP(a; \lambda)$  relative to some rectangular Cartesian axes in  $P(a; \lambda)$  satisfy a second degree equation. Hence  $SP(a; \lambda)$  is a quadric surface in  $P(a; \lambda)$ . The nature of this

surface depends, naturally, on  $a$  and  $\lambda$ . If  $\lambda \neq 0$ , then dividing the equation of the plane  $P(a; \lambda)$  by  $\lambda$ , we see that the planes  $P(a; \lambda)$  and  $P(\frac{1}{\lambda}a; 1)$  coincide. Hence we need consider only the special surface  $SP(a; 1)$ .

Let  $a$  be nonsingular. If  $x \in SP(a; 1)$  then  $\text{tr}(ax) = 1$  and so  $ax \in E(1)$ . Hence  $x \in a^{-1}E(1)$ . The converse is obvious and so  $SP(a; 1) = a^{-1}E(1)$ . Since  $E(1)$  is a hyperboloid of one sheet and since the map  $x \mapsto a^{-1}x$  is an affine map of  $P(I; 1)$  onto  $P(a; 1)$ ,  $SP(a; 1)$  must be a hyperboloid of one sheet.

**Proposition 4.** *If  $a$  is nonsingular  $SP(a; 1)$  is a hyperboloid of one sheet.*

Any regular semigroup  $S$  is equipped with a natural partial order [7] defined by

$$x \leq y \quad \text{if and only if} \quad R_x \leq R_y \text{ and } x = fy \text{ for some } f \in E(R_x).$$

If  $a \in M_2$  is nonsingular then, it can be shown that  $SP(a; 1)$  is the set of nonzero elements in  $S_2$  which are less than  $a$  under the natural partial order in  $M_2$ .

Let  $a \in S_2$ . If  $x \in S_2$  is an inverse of  $a$  then we have  $axa = a$  and so  $ax \in E(1)$  which implies that  $x \in SP(a; 1)$ . The converse also holds. Thus, if  $0 \neq a \in S_2$ , then  $SP(a; 1)$  is the set of inverses of  $a$ . To understand the geometry of  $SP(a; 1)$ , we consider the special case  $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . Clearly,  $x \in P(e; 1)$  if and only if  $x_1 = 1$ . Let  $O', A, B, C$  denote  $(1, 0, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 0, 1, 0)$  and  $(1, 0, 0, 1)$  respectively. Then the vectors  $\overrightarrow{O'A}, \overrightarrow{O'B}, \overrightarrow{O'C}$  form an orthonormal basis in  $P(e; 1)$ . If  $Q$  denotes  $(1, x_2, x_3, x_4)$  then

$$\overrightarrow{O'Q} = x_2 \overrightarrow{O'A} + x_3 \overrightarrow{O'B} + x_4 \overrightarrow{O'C}$$

so that the coordinates  $(X, Y, Z)$  of  $Q$  relative to this system of axes are  $X = x_2$ ,  $Y = x_3$ ,  $Z = x_4$ . Now  $x$  is in  $SP(e; 1)$  if and only if  $x_1 = 1$  and  $x_4 = x_2x_3$  and so  $SP(e; 1)$  is determined by the cartesian equation  $XY = Z$  which is the equation of a hyperbolic paraboloid. This argument can be modified for arbitrary  $a \in S_2$  to yield the following result.

**Proposition 5.** *Let  $a$  be singular. Then  $SP(a; 1)$  is the set of inverses of  $a$  and is a hyperbolic paraboloid.*

We next consider intersections of  $S_2$  with hyperplanes passing through the origin. These are sets of the form  $SP(a; 0)$ . If  $a$  is nonsingular then  $SP(a; 0) = a^{-1}SP(I; 0)$ . Since  $SP(I; 0)$  is a cone with vertex at the origin,  $SP(a; 0)$  is also a cone. If  $a$  is singular and nonzero then using some technical arguments, we can show that  $SP(a; 0) \setminus \{0\}$  is the union of an  $\mathcal{L}$ -class and an  $\mathcal{R}$ -class in  $S_2$ , that is, a union of two punctured planes.

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